End-Semester Exam ISI Bangalore May 2019

Introduction to Algebraic Geometry 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) There are a total of 110 points in the paper. You will be awarded a maximum of 100.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(c) By default, k denotes an algebraically closed field and $\mathbb{A}^n_k$ is the affine n-space over k while $\mathbb{P}^n_k$ is the projective n-space over k. By default, the polynomial ring of functions on $\mathbb{A}^n_k$ is denoted as $k[x_1, \ldots, x_n]$ while for $n = 1, 2, 3$ we also use the usual notation of x, y, z for the variables.
(d) We will use $\mathcal{V}(-)$ to denote the common zero locus (in suitable affine or projective space) of any collection of polynomials and $\mathcal{I}(-)$ the ideal of functions vanishing on a given subset of affine or projective space.

---

1. [10 points] Give an example of a map of affine algebraic sets $f\colon X \to Y$ such that the image of X in Y is dense but neither open nor closed.

2. [20 points] Let X be the open subset of $\mathbb{P}^2_k$ obtained by removing one point from $\mathbb{P}^2_k$.
   (i) Is X connected ?
   (ii) Is X quasi-compact ?
   (iii) Find an affine open cover of X (i.e., a covering by open subsets which are isomorphic to affine algebraic sets) and prove that the ring of globally regular functions $\mathcal{O}[X]$ is constant.

3. [20 points] In each of the following examples, for the given plane curve C, find a surjective birational map $f\colon \widetilde{C} \to C$ such that $\widetilde{C}$ is also affine and its coordinate ring $k[\widetilde{C}]$ is integrally closed in its field of fractions.
   (i) $C = \mathcal{V}(y^2 - x^3)$
   (ii) $C = \mathcal{V}(y^2 - x^2(x+1))$.

4. [20 points] Consider the regular map $\phi\colon \mathbb{P}^1 \to \mathbb{P}^4$ given by $[a : b] \to [a^4 : a^3b : a^2b^2 : ab^3 : b^4]$. Let C denote the image of ϕ.
   (i) Prove that C is a closed subset of $\mathbb{P}^4$.
   (ii) Verify that ϕ induces a bijection from $\mathbb{P}^1$ to C.
   (iii) Verify that the inverse map $C \to \mathbb{P}^1$ is regular to deduce that $\phi\colon \mathbb{P}^1 \to C$ is an isomorphism

5. [20 points] Let $k[x, y, z, w]$ denote the homogeneous coordinate ring of $\mathbb{P}^3$. Construct a non-constant regular map from the variety $\mathcal{V}(xy - zw)$ in $\mathbb{P}^3$ to $\mathbb{P}^1$.

6. [20 points] State and give a proof of Noether normalization lemma.